A note on the pressure of strong solutions to the Stokes system in bounded and exterior domains

Jörg Wolf
Department of Mathematics
Humboldt University Berlin
joerg@math.hu-berlin.de

Abstract

We consider the Stokes problem in an exterior domain $\Omega \subset \mathbb{R}^n$ with an external force $\boldsymbol{f} \in L^s(0,T;\boldsymbol{W}^{k,r}(\Omega))$ $(k \in \mathbb{N}, 1 < r < \infty)$. In the present paper we show that in contrast to $\boldsymbol{u}$ the boundary regularity of the pressure can be improved according to the differentiability of $\boldsymbol{f}$ up to order k. In particular, this implies that the pressure is smooth with respect to $x \in \Omega$ if $\boldsymbol{f}$ is smooth with respect to $x \in \Omega$.

Keywords Stokes equations, exterior domain, boundary regularity

Mathematics subject classification 35Q30, 76D03.

1 Introduction

Let $\Omega \subset \mathbb{R}^n$ $(n \in \mathbb{N}, n \geq 2)$ be an exterior domain, i. e. $\mathbb{R}^n \setminus \overline{\Omega}$ is a bounded domain in $\mathbb{R}^n$. Let $0 < T < +\infty$. Set $Q = \Omega \times (0,T)$. In the present paper we consider the Stokes problem

$$\operatorname{div} \boldsymbol{u} = 0 \quad \text{in} \quad Q \tag{1.1}$$

$$\partial_t \boldsymbol{u} - \Delta \boldsymbol{u} = -\nabla p + \boldsymbol{f} \quad \text{in} \quad Q, \tag{1.2}$$

$$\boldsymbol{u} = 0 \quad \text{on} \quad \partial\Omega \times (0,T), \qquad \lim_{|x|\to\infty} \boldsymbol{u}(x,\cdot) = 0, \tag{1.3}$$

$$\boldsymbol{u}(0) = 0 \quad \text{in} \quad \Omega, \tag{1.4}$$

where $\boldsymbol{u} = (u^1, \ldots, u^n)$ denotes the unknown velocity of the fluid, p the unknown pressure and $\boldsymbol{f}$ the given external force. The Stokes problem has been extensively studied in the past. In particular, for the case Ω is the half space or an C^2 domain with compact boundary the L^p-theory is well-known. Based on potential theory in [14] Solonnikov proved that for every $\boldsymbol{f} \in \boldsymbol{L}^q(Q)$ there exists a unique solution $(\boldsymbol{u}, p)$ to (1.1)–(1.4) such that $\partial_t \boldsymbol{u}, \nabla^2 \boldsymbol{u} \in \boldsymbol{L}^q(Q)$, and $\nabla p \in \boldsymbol{L}^p(Q)$. By using the semi group approach, similar results have been obtained in [5], [6], [3]. For the corresponding estimates on the pressure we refer to [13]. An optimal result for the anisotropic case when $\boldsymbol{f}$ belongs to $L^s(0,T;\boldsymbol{L}^q(\Omega))$ has been proved in [7] for the cases $\Omega = \mathbb{R}^n$, $\Omega = \mathbb{R}^n_+$, and a C^2 domain Ω with compact boundary.

By standard arguments from the regularity theory of parabolic equations one gets the regularity $\boldsymbol{u}$ and p in dependence of the regularity of the right-hand side $\boldsymbol{f}$ in time and space. However, if $\boldsymbol{f}$ is only smooth in $x \in \Omega$ it is not clear whether $\boldsymbol{u}$ is smooth in x up to the boundary. In the present paper we will see that such a property at least holds for the pressure p, which

is due to the fact that $\Delta p = 0$ if div $\boldsymbol{f} = 0$. More precisely, the condition$\boldsymbol{f} \in L^s(0,T;\boldsymbol{W}^{k,\,q}(\Omega))$ $(1 < s, q, < +\infty; k \in \mathbb{N})$ implies $\nabla p \in L^s(0,T;\boldsymbol{W}^{k,\,q}(\Omega))$. Note that our result relies essentially on the fact that the initial data is zero. In general our result may not be true as there is a counter-example obtained in [9]. More precisely, there exists an initial data, and a solution $\boldsymbol{u}, p$ to the Stokes system such that $\|\nabla \boldsymbol{u}(t)\|_{\boldsymbol{L}^2}$ is continuous as $t \to 0^+$, while the corresponding estimate on the pressure $\|p(t)\|_{L^2}$ may blow up as $t \to 0^+$.

First we shall introduce the basic notations regarding the function spaces used throughout the paper. By $W^{k,\,q}(\Omega), W^{k,\,q}_0(\Omega)$ we denote the usual Sobolev spaces. Vector functions and spaces of vector valued functions will be denoted by bold face letters, i. e. we write $\boldsymbol{L}^q(\Omega), \boldsymbol{W}^{k,\,q}(\Omega)$, etc. instead of $L^q(\Omega;\mathbb{R}^n), W^{k,\,q}(\Omega;\mathbb{R}^n)$, etc. In addition, we use the following spaces of solenoidal functions

$$\begin{aligned}
\boldsymbol{L}^q_\sigma(\Omega) &= \text{ closure of } \mathscr{C}^\infty_{0,\sigma}(\Omega) \text{ w.r.t. the norm } \|\cdot\|_{L^q}\\
\boldsymbol{W}^{k,\,q}_{0,\sigma}(\Omega) &= \text{ closure of } \mathscr{C}^\infty_{0,\sigma}(\Omega) \text{ w.r.t. the norm } \|\cdot\|_{W^{k,q}},
\end{aligned}$$

where $\mathscr{C}^\infty_{0,\sigma}(\Omega)$ stands for the space of all smooth solenoidal vector fields with compact support in Ω. Given a Banach space X by $L^q(0,T;X)$ we denote the space of Bochner measurable functions $f : (0,T) \to X$ such that

$$\begin{aligned}
\|f\|^q_{L^q(0,T;X)} &= \int_0^T \|f(t)\|^q_X dt < +\infty \quad \text{if} \quad 1 \le q < +\infty,\\
\|f\|_{L^\infty(0,T;X)} &= \operatorname*{ess\,sup}_{t\in(0,T)} \|f(t)\|_X < +\infty \quad \text{if} \quad q = +\infty.
\end{aligned}$$

Now, let us introduce the notion of a strong solution to (1.1)–(1.4).

Definition 1.1 Let $\boldsymbol{f} \in L^s(0,T;\boldsymbol{L}^q(\Omega))$ $(1 < s, q < +\infty)$. A pair $(\boldsymbol{u}, p)$ is called a *strong solution* to (1.1)–(1.4) if $\boldsymbol{u} \in L^s(0,T;\boldsymbol{W}^{1,\,q}_{0,\sigma}(\Omega)), p \in L^s(0,T;L^1_{\text{loc}}(\overline{\Omega}))$ and

$$\partial_i\partial_j \boldsymbol{u}, \partial_t \boldsymbol{u}, \nabla p \in L^s(0,T;\boldsymbol{L}^q(\Omega)), \quad i,j = 1,2,3,$$

such that (1.1), (1.2) holds a. e. in Q, while (1.4) is fulfilled such that $\boldsymbol{u} = 0$ a. e. in $\Omega \times \{0\}$.

For the existence of a strong solution to (1.1)–(1.4) cf. in [7].

Our main result is the following

Theorem 1 *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain or a bounded domain with $\partial\Omega \in C^{2+k}$ $(k \in \mathbb{N})$. For $\boldsymbol{f} \in L^s(0,T;\boldsymbol{W}^{k,\,q}(\Omega))$ $(1 < s, q < +\infty; k \in \mathbb{N})$, let $(\boldsymbol{u}, p)$ be the strong solution to* (1.1)–(1.4). *Then,*

$$\nabla p \in L^s(0,T;\boldsymbol{W}^{k,\,q}(\Omega)).$$

In addition, there holds

$$\|\nabla^{k+1} p\|_{L^s(0,T;\boldsymbol{L}^q(\Omega))} \le c\|\boldsymbol{f}\|_{L^s(0,T;\boldsymbol{W}^{k,\,q}(\Omega))}, \tag{1.5}$$

where $c = \text{const} > 0$ depending only on s, q, k and the geometric properties of $\partial\Omega$.

2 Remarks on the equation $\operatorname{div} \boldsymbol{v} = f$

Let $G \subset \mathbb{R}^n$ be a bounded domain, star-shaped with respect to a ball B_R. It is well known that for all $f \in L^q(G)$ with $(f)_G = 0$ [1)] the equation $\operatorname{div} \boldsymbol{v} = f$ has a solution $\boldsymbol{v} \in \boldsymbol{W}_0^{1,q}(G)$ such that

$$\|\nabla \boldsymbol{v}\|_{L^q(G)} \le c\|f\|_{L^q(G)}$$

with $c = \text{const} > 0$, depending on n, q and G (cf. [2], [8]). In fact, the constant c depends on the geometric property of G, namely the ratio of G which is defined by

$$\operatorname{ratio}(G) := \frac{R_a(G)}{R_i(G)},$$

where

$$R_a(G) = \inf\{R > 0 \,|\, \exists B_R(x_0) : G \subset B_R(x_0)\},$$
$$R_i(G) = \sup\{r > 0 \,|\, \exists B_r(x_0) : G \text{ is star-shaped w.r.t } B_r(x_0)\}.$$

For instance $\operatorname{ratio}(G) = 1$ if G is a ball, and $\operatorname{ratio}(G) = \sqrt{n}$ if G is a cube. Moreover, the ratio is invariant under translation and scaling, i. e.

$$\operatorname{ratio}(\lambda G) = \operatorname{ratio}(G) \quad \forall\, \lambda > 0.$$

Now, let G such that $2 < R_i(G) < 3$. In particular, G is star shaped with respect to a ball $B_2 = B_2(x_0)$. Without loss of generality we may assume that $x_0 = 0$. Let $\phi \in C_0^\infty(B_2)$. We define

$$\mathscr{B}_\phi f(x) = \int\limits_{\mathbb{R}^n} f(x-y)\boldsymbol{K}_\phi(x,y)dy, \quad x \in \mathbb{R}^n, \quad f \in C_0^\infty(G),$$

where

$$\boldsymbol{K}_\phi(x,y) = \frac{y}{|y|^n}\int\limits_0^\infty \phi\Big(x + r\frac{y}{|y|^n}\Big)(|y|+r)^{n-1}dr, \quad (x,y) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}).$$

As in [2], [8] it has been proves that $\mathscr{B}_\phi f \in \boldsymbol{C}_0^\infty(G)$ for all $f \in C_0^\infty(G)$. In addition, there holds

$$\|\nabla^k \mathscr{B}_\phi f\|_{\boldsymbol{L}^q(G)} \le c\|\nabla^{k-1} f\|_{L^q(G)} \quad \forall\, f \in C_0^\infty(G) \tag{2.1}$$

with a constant depending on n, k, q, ϕ and $\operatorname{ratio}(G)$ only. Furthermore, there holds

$$\operatorname{div} \mathscr{B}_\phi f = f\int\limits_{B_1} \phi(y)dy - \phi\int\limits_G f(y)dy \quad \text{in} \quad G. \tag{2.2}$$

In particular, if $\int\limits_{B_1} \phi(y)dy = 1$ and $\int\limits_G f(y)dy = 0$ then $\boldsymbol{v} = \mathscr{B}_\phi f$ solves the equation $\operatorname{div} \boldsymbol{v} = f$. Finally, by (2.1) we may extend $\mathscr{B}_\phi$ to an operator $\mathscr{L}(W^{k-1,q}(G), \boldsymbol{W}^{k,q}(G))$ denoted again by $\mathscr{B}_\phi$.

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1) Let $A \subset \mathbb{R}^n$ be a measurable set with $\operatorname{mes}(A)$. Given $v \in L^1(A)$ by $(v)_A$ we denote the mean value $\frac{1}{\operatorname{mes}(A)}\int\limits_A v(x)dx$.

Let $i, j \in \{1, \dots, n\}$. Observing, that

$$\begin{aligned} \partial_j \mathscr{B}_\phi(\partial_i f) &= \partial_i \partial_j \mathscr{B}_\phi(f) - \partial_j \mathscr{B}_{\partial_i \phi}(f) \quad \text{in} \quad G, \\ \partial_i \partial_j \mathscr{B}_\phi(f) &= \partial_i \mathscr{B}_\phi(\partial_j f) + \partial_i \mathscr{B}_{\partial_j \phi}(f) \quad \text{in} \quad G, \end{aligned}$$

we see that

$$\partial_j \mathscr{B}_\phi(\partial_i f) = \partial_i \mathscr{B}_\phi(\partial_j f) + \partial_i \mathscr{B}_{\partial_j \phi}(f) - \partial_j \mathscr{B}_{\partial_i \phi}(f) \quad \text{in} \quad G.$$

By the aid of (2.1), and Poincaré's inequality, using the above identity, we get

$$\|\partial_j \mathscr{B}_\phi(\partial_i f)\|_{\boldsymbol{L}^q(G)} \le c(\|\partial_j f\|_{L^q(G)} + \|f\|_{L^q(G)}) \quad \forall\, f \in W_0^{1,q}(G), \tag{2.3}$$

$$\begin{cases} \|\nabla^2 \partial_j \mathscr{B}_\phi(\partial_i f)\|_{\boldsymbol{L}^q(G)} \le c(\|\partial_i \nabla_* \nabla f\|_{\boldsymbol{L}^q(G)} + \|\partial_j \partial_n \partial_n f\|_{L^q(G)} + \|\nabla^2 f\|_{\boldsymbol{L}^q(G)})\,^{2)} \\ \forall\, f \in W_0^{3,q}(G), \end{cases} \tag{2.4}$$

where $c = \text{const} > 0$, depending on n, q and ratio(G).

Now, let G be a bounded domain, star-shaped with respect to a ball B. Let $R := \frac{1}{2} R_i(G)$. Thus, there exist $B_R(x_0)$ such that G is star shaped to the ball $B_R(x_0)$. Without loss of generality we may assume that $x_0 = 0$. Let $\phi \in C_0^\infty(B_1)$ with $\int\limits_{B_1} \phi(y) dy = 1$. We define $\mathscr{B} : W_0^{k-1,q}(G) \to \boldsymbol{W}_0^{k,q}(G)$ by setting

$$\mathscr{B}(f)(x) = R \mathscr{B}_\phi(\widetilde{f})\Big(\frac{x}{R}\Big), \quad x \in G, \quad f \in W_0^{k-1,q}(G),$$

where $\widetilde{f}(y) = f(Ry)\,(y \in R^{-1}G)$. Using the transformation formula of the Lebesgue integral, in view of (2.1), we see that

$$\begin{aligned} \|\nabla^k \mathscr{B}(f)\|_{\boldsymbol{L}^q(G)} &= R^{n/q-k+1} \|\nabla^k \mathscr{B}_\phi(\widetilde{f})\|_{\boldsymbol{L}^q(R^{-1}G)} \le c R^{n/q-k+1} \|\nabla^{k-1} \widetilde{f}\|_{\boldsymbol{L}^q(R^{-1}G)} \\ &= c \|\nabla^{k-1} f\|_{\boldsymbol{L}^q(G)}, \end{aligned} \tag{2.5}$$

where $c = \text{const} > 0$ depends on n, q and ratio$(R^{-1}G) =$ ratio(G). In addition, from (2.3), and (2.4) we deduce

$$\|\partial_j \mathscr{B}(\partial_i f)\|_{\boldsymbol{L}^q(G)} \le c(\|\partial_j f\|_{L^q(G)} + R^{-1} \|f\|_{L^q(G)}) \quad \forall\, f \in W_0^{1,q}(G), \tag{2.6}$$

$$\begin{cases} \|\nabla^2 \partial_j \mathscr{B}(\partial_i f)\|_{\boldsymbol{L}^q(G)} \le c(\|\partial_i \nabla_* \nabla f\|_{\boldsymbol{L}^q(G)} + \|\partial_j \partial_n \partial_n f\|_{L^q(G)} + R^{-1} \|\nabla^2 f\|_{\boldsymbol{L}^q(G)}) \\ \forall\, f \in W_0^{3,q}(G), \end{cases} \tag{2.7}$$

$(i, j = 1, \dots, n)$ with a constant c, depending on n, q and ratio(G) only. Furthermore, from (2.2) we get

$$\operatorname{div} \mathscr{B}(f)(x) = f(x) - \phi\Big(\frac{x}{R}\Big) R^{-n} \int\limits_G f(y) dy \quad \text{for a.e. } x \in G. \tag{2.8}$$

[2)] Here ∇_* denotes the reduced gradient $(\partial_1, \dots, \partial_{n-1})$.

3 Proof of Theorem 1

Proof 1° By decomposing the right-hand side into a solenoidal field, and a gradient field, we are able to reduce the problem to the case $\operatorname{div} \boldsymbol{f} = 0$. Let $\boldsymbol{E} : \boldsymbol{W}^{k,q}(\Omega) \to \boldsymbol{W}^{k,q}(\mathbb{R}^n)$ denote an extension operator such that

$$\|\boldsymbol{E}\boldsymbol{v}\|_{\boldsymbol{W}^{k,q}(\mathbb{R}^n)} \le c\|\boldsymbol{v}\|_{\boldsymbol{W}^{k,q}(\Omega)} \quad \forall \boldsymbol{v} \in \boldsymbol{W}^{k,q}(\Omega).$$

Let $\boldsymbol{P} : \boldsymbol{W}^{k,q}(\mathbb{R}^n) \to \boldsymbol{W}^{k,q}_{0,\sigma}(\mathbb{R}^n)$ denote the Helmholtz-Leray projection. Given $\boldsymbol{v} \in \boldsymbol{W}^{k,q}(\Omega)$ we have

$$\boldsymbol{v} = \boldsymbol{P}\boldsymbol{E}\boldsymbol{v} + (I - \boldsymbol{P})\boldsymbol{E}\boldsymbol{v} \quad \text{a.e. in } \Omega.$$

In addition, there exists a constant $c > 0$ depending only on n, q, k and Ω such that

$$\|\boldsymbol{P}\boldsymbol{E}\boldsymbol{v}\|_{\boldsymbol{W}^{k,q}(\Omega)} + \|(I - \boldsymbol{P})\boldsymbol{E}\boldsymbol{v}\|_{\boldsymbol{W}^{k,q}(\Omega)} \le c\|\boldsymbol{v}\|_{\boldsymbol{W}^{k,q}(\Omega)} \quad \forall \boldsymbol{v} \in \boldsymbol{W}^{k,q}(\Omega). \tag{3.1}$$

Now, for $\boldsymbol{f} \in L^s(0,T; \boldsymbol{W}^{k,r}(\Omega))$ let $(\boldsymbol{u}, p)$ be a strong solution to (1.1)–(1.4). Observing $I - \boldsymbol{P} = \nabla(\Delta^{-1} \operatorname{div})$ recalling the definition of $\boldsymbol{E}$ we get

$$\boldsymbol{E}\boldsymbol{f} = \boldsymbol{P}\boldsymbol{E}\boldsymbol{f} + (I - \boldsymbol{P})\boldsymbol{E}\boldsymbol{f} = \boldsymbol{P}\boldsymbol{E}\boldsymbol{f} + \nabla(\Delta^{-1} \operatorname{div} \boldsymbol{E}\boldsymbol{f}) \quad \text{a.e. in } Q.$$

Since, $\nabla(\Delta^{-1} \operatorname{div} \boldsymbol{E}\boldsymbol{f}) = (\Delta^{-1}\nabla \operatorname{div} \boldsymbol{E}\boldsymbol{f}) \in L^s(0,T; \boldsymbol{W}^{k,q}(\mathbb{R}^n))$ we see that $\boldsymbol{P}\boldsymbol{E}\boldsymbol{f} \in L^s(0,T; \boldsymbol{W}^{k,q}(\mathbb{R}^n))$. Thus, we can replace $\boldsymbol{f}$ by the restriction of $\boldsymbol{P}\boldsymbol{E}\boldsymbol{f}$ on Q, and p by the restriction of $-\Delta^{-1} \operatorname{div} \boldsymbol{E}\boldsymbol{f} + p$ on Q. Hence, in what follows without loss of generality we may assume that

$$\operatorname{div} \boldsymbol{f} = 0, \quad \text{and} \quad \Delta p = 0 \quad \text{a.e. in } \ Q. \tag{3.2}$$

2° Secondly, we recall a well-known result by Giga and Sohr [7] which is the following

Lemma 3.1 *Let $\Omega = \mathbb{R}^n$, $\Omega = \mathbb{R}^n_+$, Ω bounded or Ω an exterior domain with $\partial\Omega \in C^2$. For every $\boldsymbol{g} \in L^s(0,T; \boldsymbol{L}^q_\sigma(\Omega))$ $(1 < s, q, < +\infty)$ there exists a unique solution $(\boldsymbol{v}, \pi) \in L^s(0,T; \boldsymbol{W}^{2,q}_{\mathrm{loc}}(\Omega)) \times L^s(0,T; W^{1,q}_{\mathrm{loc}}(\Omega))$ to the Stokes problem*

$$\begin{aligned} \partial_t \boldsymbol{v} - \Delta \boldsymbol{v} = -\nabla\pi + \boldsymbol{g} \quad &\textit{and} \quad \operatorname{div} \boldsymbol{v} = 0 \quad \textit{in} \quad \Omega \times (0,T), \\ \boldsymbol{v} = 0 \quad &\textit{on} \quad \partial\Omega \times (0,T), \\ \boldsymbol{v}(0) = 0 \quad &\textit{on} \quad \Omega \times \{0\}, \end{aligned}$$

such that $\partial_t \boldsymbol{v}, \partial_i\partial_j \boldsymbol{v}, \nabla\pi \in L^s(0,T; \boldsymbol{L}^q(\Omega))$ $(i, j = 1, \ldots, n)$, and there holds,

$$\|\partial_t \boldsymbol{v}\|_{L^s(0,T;\boldsymbol{L}^q(\Omega))} + \|\nabla^2 \boldsymbol{v}\|_{L^s(0,T;\boldsymbol{L}^q(\Omega))} + \|\nabla\pi\|_{L^s(0,T;\boldsymbol{L}^q(\Omega))} \le c\|\boldsymbol{g}\|_{L^s(0,T;\boldsymbol{L}^q(\Omega))}, \tag{3.3}$$

where the constant c depends only on n, s, q and Ω.

As a consequence of Lemma 3.1 we get the existence of a unique solution $(\boldsymbol{u}, p) \in L^s(0,T; \boldsymbol{W}^{2,q}_{\mathrm{loc}}(\Omega)) \times L^s(0,T; W^{1,q}_{\mathrm{loc}}(\Omega))$ to the Stokes system (1.1)–(1.4), such that

$$\|\partial_t \boldsymbol{u}\|_{L^s(0,T;\boldsymbol{L}^q(\Omega))} + \|\nabla^2 \boldsymbol{u}\|_{L^s(0,T;\boldsymbol{L}^q(\Omega))} + \|\nabla p\|_{L^s(0,T;\boldsymbol{L}^q(\Omega))} \le c\|\boldsymbol{f}\|_{L^s(0,T;\boldsymbol{L}^q(\Omega))}. \tag{3.4}$$

3° *Local estimates* We restrict ourself to case that Ω is an exterior domain. The opposite case can be treated in a similar way. Clearly, $G := \mathbb{R}^n \setminus \overline{\Omega}$ is a bounded domain. Let G', G'' are

bounded open sets such that $\overline{G} \subset G'$ and $\overline{G'} \subset G''$. Set $\Omega'' = \mathbb{R}^3 \setminus \overline{G''}$ and $\Omega' = \mathbb{R}^3 \setminus \overline{G'}$. Then, let $\zeta \in C^\infty(\mathbb{R}^3)$ denote a cut-off function such that $\zeta \equiv 1$ on Ω'', and $\zeta \equiv 0$ in G'. In particular, $\operatorname{supp}(\nabla\zeta) \subset G'' \setminus G'$. Observing $\operatorname{div}(\boldsymbol{u}(t)\zeta) = \boldsymbol{u}(t)\cdot\nabla\zeta$, it follows that $\operatorname{supp}(\boldsymbol{u}(t)\cdot\nabla\zeta) \subset\subset G''\setminus G'$ for a. e. $t \in (0,T)$.

Next, let $1 < R < +\infty$ such that $G'' \subset B_R$. By $\mathscr{B} : W_0^{k-1,q}(B_R) \to \boldsymbol{W}_0^{k,q}(B_R)$ we denote the Bogowskiĭ operator defined in Section 2. We now define

$$\boldsymbol{z}(t) = \mathscr{B}(\boldsymbol{u}(t)\cdot\nabla\zeta), \quad t \in [0,T).$$

Let $t \in (0,t)$. Since $\int\limits_{B_R} \boldsymbol{u}(t)\cdot\nabla\zeta dx = 0$, in view of (2.8) we have

$$\operatorname{div}\boldsymbol{z}(t) = \boldsymbol{u}(t)\cdot\nabla\zeta \quad \text{a. e. in } B_R.$$

Thanks to (2.6), recalling that $\operatorname{ratio}(B_R) = 1$, there exists a constant $c > 0$ depending only on q and n such that

$$\|\boldsymbol{z}(t)\|_{\boldsymbol{W}^{3,q}(B_R)} \le c\|\boldsymbol{u}(t)\cdot\nabla\zeta\|_{W^{2,q}(B_R)} \quad \text{for a. e. } t \in (0,T).$$

Making use of the embedding $\boldsymbol{W}_0^{3,q}(B_R) \hookrightarrow \boldsymbol{W}^{3,q}(\mathbb{R}^n)$ the above inequality implies that $\boldsymbol{z} \in L^s(0,T;\boldsymbol{W}^{3,q}(\mathbb{R}^n))$. Together with (3.4), and the Sobolev-Poincaré inequality we obtain

$$\|\boldsymbol{z}\|_{L^s(0,T;\boldsymbol{W}^{3,q}(\mathbb{R}^n))} \le c\|\boldsymbol{u}\|_{L^s(0,T;\boldsymbol{W}^{2,q}(\Omega\cap B_R))} \le c\|\boldsymbol{f}\|_{L^s(0,T;\boldsymbol{L}^q(\Omega))}. \tag{3.5}$$

By an analogous reasoning taking into account $\partial_t\boldsymbol{z} = \mathscr{B}(\partial_t\boldsymbol{u}\cdot\nabla\zeta)$ a. e. in $\mathbb{R}^n \times (0,T)$ we see that $\partial_t\boldsymbol{z} \in L^s(0,T;\boldsymbol{W}^{1,q}(\mathbb{R}^n))$. In addition, by virtue of (3.4) we obtain

$$\|\partial_t\boldsymbol{z}\|_{L^s(0,T;\boldsymbol{W}^{1,q}(\mathbb{R}^3))} \le c\|\partial_t\boldsymbol{u}\|_{L^s(0,T;\boldsymbol{L}^q(\Omega))} \le c\|\boldsymbol{f}\|_{L^s(0,T;\boldsymbol{L}^q(\Omega))}. \tag{3.6}$$

Next, let $k \in \{1,\ldots,n\}$ be fixed. We define

$$\begin{cases} \boldsymbol{v}(x,t) = \partial_k(\boldsymbol{u}(x,t)\zeta(x) - \boldsymbol{z}(x,t)), & (x,t) \in (G''\setminus G')\times(0,T), \\ \boldsymbol{v}(x,t) = -\partial_k\boldsymbol{z}(x,t), & (x,t) \in (\mathbb{R}^n\setminus(G''\setminus G'))\times(0,T), \end{cases}$$

and

$$\begin{cases} \pi(x,t) = \partial_k(p(x,t)\zeta(x)), & (x,t) \in (G''\setminus G')\times(0,T), \\ \pi(x,t) = 0, & (x,t) \in (\mathbb{R}^n\setminus(G''\setminus G'))\times(0,T). \end{cases}$$

Then the pair $(\boldsymbol{v},\pi)$ solves the Stokes system

$$\begin{aligned} \operatorname{div}\boldsymbol{v} &= 0 && \text{in } \mathbb{R}^n\times(0,T), \\ \partial_t\boldsymbol{v} - \Delta\boldsymbol{v} &= -\nabla\pi + \boldsymbol{g} && \text{in } \mathbb{R}^n\times(0,T), \\ \boldsymbol{v} &= 0 && \text{on } \mathbb{R}^n\times\{0\}, \end{aligned}$$

where

$$\begin{aligned} \boldsymbol{g} = (p - p_{B_R})\nabla\zeta &- 2\partial_k(\nabla\boldsymbol{u}\cdot\nabla\zeta) - \partial_k(\boldsymbol{u}\Delta\zeta) \\ &- \partial_k\partial_t\boldsymbol{z} + \partial_k\Delta\boldsymbol{z} + \partial_k(\boldsymbol{f}\zeta) \qquad \text{a. e. in } \mathbb{R}^n\times(0,T). \end{aligned}$$

In view of (3.3), (3.5), and (3.6) we see that $\boldsymbol{g} \in L^s(0,T;\boldsymbol{L}^q(\mathbb{R}^n))$. In addition, there holds

$$\|\boldsymbol{g}\|_{L^s(0,T;\boldsymbol{L}^q(\mathbb{R}^n))} \le c\|\boldsymbol{f}\|_{L^s(0,T;\boldsymbol{W}^{1,q}(\Omega))}.$$

Thus, applying Lemma 3.1 with $\Omega = \mathbb{R}^n$, and using the last inequality we see that

$$\begin{aligned}
&\|\partial_t \boldsymbol{v}\|_{L^s(0,T;\boldsymbol{L}^q(\mathbb{R}^n))} + \|\nabla^2 \boldsymbol{v}\|_{L^s(0,T;\boldsymbol{L}^q(\mathbb{R}^n))} + \|\nabla \pi\|_{L^s(0,T;\boldsymbol{L}^q(\mathbb{R}^n))}\\
&\qquad \le c\|\boldsymbol{g}\|_{L^s(0,T;\boldsymbol{L}^q(\mathbb{R}^n))}\\
&\qquad \le c\|\boldsymbol{f}\|_{L^s(0,T;\boldsymbol{W}^{1,q}(\Omega))}.
\end{aligned}$$

Recalling the definition of $\boldsymbol{v}$, making use of (3.5), (3.6), and (3.4), we infer from above

$$\begin{aligned}
&\Big\|\zeta\partial_t\partial_k \boldsymbol{u}\|_{L^s(0,T;\boldsymbol{L}^q(\Omega))} + \|\zeta\nabla^2\partial_k \boldsymbol{u}\|_{L^s(0,T;\boldsymbol{L}^q(\Omega))} + \|\zeta\nabla\partial_k p\|_{L^s(0,T;\boldsymbol{L}^q(\Omega))}\\
&\qquad \le c\|\boldsymbol{f}\|_{L^s(0,T;\boldsymbol{W}^{1,q}(\Omega))}.
\end{aligned}$$

Iterating the above argument k times, we get

$$\begin{aligned}
&\|\partial_t \boldsymbol{u}\|_{L^s(0,T;\boldsymbol{W}^{k,q}(\Omega'))} + \|\boldsymbol{u}\|_{L^s(0,T;\boldsymbol{W}^{k+2,q}(\Omega'))} + \|\nabla p\|_{L^s(0,T;\boldsymbol{W}^{k,q}(\Omega'))}\\
&\qquad \le c\|\boldsymbol{f}\|_{L^s(0,T;\boldsymbol{W}^{k,q}(\Omega))}
\end{aligned} \tag{3.7}$$

($k \in \mathbb{N}$), where $c = \text{const} > 0$, depending on s, q, k, and Ω only.

4° *Boundary regularity* Let $x_0 \in \partial\Omega$. Up to translation and rotation we may assume that $x_0 = 0$ and $\boldsymbol{n}(0) = -\boldsymbol{e}_n$, where $\boldsymbol{n}(0)$ denotes the outward unite normal on Ω at x_0. According to our assumption on the boundary of Ω there exists $0 < R < +\infty$, and $h \in C^{2+k}(B'_R)$ such that

(i) $\partial\Omega \cap (B'_R \times (-R,R)) = \{(y', h(y')); y' \in B'_R\}$;

(ii) $\{(y', y_n); y' \in B'_R, h(y') < y_n < h(y') + R\} \subset \Omega$;

(iii) $\{(y', y_n); y' \in B'_R, -R + h(y') < y_n < h(y')\} \subset \Omega^c$ [3].

Set $U_R = B'_R \times (-R,R), U_R^+ = B'_R \times (0,R)$, and define $\boldsymbol{\Phi} : U_R \to \boldsymbol{\Phi}(U_R)$ by

$$\boldsymbol{\Phi}(y) = (y', h(y') + y_n)^\top, \quad y \in U_R.$$

Elementary,

$$D\boldsymbol{\Phi}(y) = \begin{pmatrix} 1 & 0 & \dots & 0 & 0\\ 0 & 1 & \dots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \dots & 1 & 0\\ \partial_1 h(y) & \partial_2 h(y) & \dots & \partial_{n-1} h(y) & 1 \end{pmatrix},$$

$$(D\boldsymbol{\Phi}(y))^{-1} = \begin{pmatrix} 1 & 0 & \dots & 0 & 0\\ 0 & 1 & \dots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \dots & 1 & 0\\ -\partial_1 h(y) & -\partial_2 h(y) & \dots & -\partial_{n-1} h(y) & 1 \end{pmatrix}.$$

[3] Here $y' = (y_1, \dots, y_{n-1}) \in \mathbb{R}^{n-1}$, and B'_R denotes the two dimensional ball $\{(y_1, \dots, y_{n-1}) : y_1^2 + \ldots + y_{n-1}^2 < R^2\}$.

For the outward unit normal at $x = \mathbf{\Phi}(y)$ we have

$$\boldsymbol{n}(x) = \boldsymbol{N}(y) = \frac{(\partial_1 h(y), \dots, \partial_{n-1} h(y), -1)}{\sqrt{1 + |\nabla h(y)|^2}}, \quad y \in B'_R \times \{0\}.$$

In addition, one calculates

$$\partial_{x_i} \circ \mathbf{\Phi} = \partial_{y_i} - (\partial_{x_i} h)\partial_{y_n} \quad \text{in} \quad U_R, \quad i = 1, \dots, n\,^{4)}. \tag{3.8}$$

We set $\boldsymbol{U} = \boldsymbol{u} \circ \mathbf{\Phi}, P = p \circ \mathbf{\Phi}$ and $\boldsymbol{F} = \boldsymbol{f} \circ \mathbf{\Phi}$ a. e. in $U_R^+ \times (0, T)$. By the aid of (3.8) we easily get

$$(\mathrm{div}_x \boldsymbol{u}) \circ \mathbf{\Phi} = \mathrm{div}_y \boldsymbol{U} - \nabla h \cdot \partial_{y_n} \boldsymbol{U} = 0, \tag{3.9}$$

$$(\Delta_x \boldsymbol{u}) \circ \mathbf{\Phi} = \Delta_y \boldsymbol{U} - 2\nabla h \cdot \nabla_y \partial_{y_n} \boldsymbol{U} + |\nabla h|^2 \partial_{y_n} \partial_{y_n} \boldsymbol{U} - (\Delta h)\partial_{y_n} \boldsymbol{U}, \tag{3.10}$$

$$(\nabla_x p) \circ \mathbf{\Phi} = \nabla_y P - (\nabla h)\partial_{y_n} P, \tag{3.11}$$

a. e. in $U_R^+ \times (0, T)$. Firstly, owing to (3.9) from the equation (1.1) we get

$$\mathrm{div}_y \boldsymbol{U} = \nabla h \cdot \partial_{y_n} \boldsymbol{U} \quad \text{a. e. in} \quad U_R^+ \times (0, T), \tag{3.12}$$

and with help of (3.10) and (3.11) the equation (1.2) turns into

$$\begin{aligned} \partial_t \boldsymbol{U} - \Delta \boldsymbol{U} = -\nabla P + (\partial_{y_n} P)\nabla h - 2\nabla h \cdot \nabla \partial_{y_n} \boldsymbol{U} + |\nabla h|^2 \partial_{y_n} \partial_{y_n} \boldsymbol{U} \\ - (\Delta h)\partial_{y_n} \boldsymbol{U} + \boldsymbol{F} \end{aligned} \tag{3.13}$$

a. e. in $U_R^+ \times (0, T)$.

Note that the assumption $\boldsymbol{n}(0) = -\boldsymbol{e}_n$ implies $\nabla h(0) = 0$. We now choose $0 < \delta < +\infty$ sufficiently small, which will be specified later. Since $\nabla h \in \boldsymbol{C}^0(U_R)$, there exists $0 < \rho < \frac{R}{2}$ such that

$$|\nabla h(y)| \le \delta \quad \forall\, y \in U_{2\rho}. \tag{3.14}$$

Let $\zeta \in C_0^\infty(U_{2\rho})$ denote a cut-off function such that $0 \le \zeta \le 1$ in $U_{2\rho}$, and $\zeta \equiv 1$ on U_ρ. We define $\widetilde{\boldsymbol{U}} : \mathbb{R}^n_+ \times (0, T) \to \mathbb{R}^n$ by

$$\widetilde{\boldsymbol{U}}(y, t) = \zeta(y)\boldsymbol{U}(y, t), \quad y \in U_{2\rho}^+ \times (0, T), \quad \widetilde{\boldsymbol{U}}(y, t) = 0 \quad \text{if} \quad y \in \mathbb{R}^n_+ \setminus U_{2\rho}^+ \times (0, T).$$

Let $\mathscr{B} : W_0^{k-1,q}(U_{2\rho}^+) \to \boldsymbol{W}^{k,q}(\mathbb{R}^n_+)$ denote the Bogowskiĭ operator defined in Section 2. We set

$$\begin{aligned} \boldsymbol{z}_1(y, t) &= \mathscr{B}(\zeta \nabla h \cdot \partial_{y_n} \boldsymbol{U})(y, t), \\ \boldsymbol{z}_2(y, t) &= \mathscr{B}(\nabla \zeta \cdot \boldsymbol{U})(y, t), \quad (y, t) \in \mathbb{R}^n_+ \times (0, T). \end{aligned}$$

Let $k \in \{1, \dots, n-1\}$ be fixed. We define

$$\begin{aligned} \boldsymbol{V}(y, t) &= \partial_k(\widetilde{\boldsymbol{U}}(y, t) - \boldsymbol{z}_1(y, t) - \boldsymbol{z}_2(y, t)), \\ \Pi(y, t) &= \partial_k(\zeta(y)P(y, t)), \end{aligned}$$

4) Since h is independent on y_n there holds $\partial_{x_n} \circ \mathbf{\Phi} = \partial_{y_n}$.

$(y,t) \in \mathbb{R}^n_+ \times (0,T)$. Observing that

$$\int\limits_{U^+_{2\rho}} \zeta \nabla h \cdot \partial_n \boldsymbol{U}(t) + \nabla \zeta \cdot \boldsymbol{U}(t) dy = \int\limits_{U^+_{2\rho}} \operatorname{div}_y \widetilde{\boldsymbol{U}}(t) dy = 0 \quad \text{for a.e. } t \in (0,T),$$

by the aid of (2.8) we calculate

$$\operatorname{div}_y \boldsymbol{V} = \partial_k \Big(\zeta \nabla h \cdot \partial_n \boldsymbol{U} + \nabla \zeta \cdot \boldsymbol{U} - \zeta \nabla h \cdot \partial_n \boldsymbol{U} - \nabla \zeta \cdot \boldsymbol{U} \Big) = 0 \tag{3.15}$$

a.e. in $\mathbb{R}^n_+ \times (0,T)$. In addition, taking into account (3.13), we find

$$\begin{aligned}
\partial_t \boldsymbol{V} - \Delta \boldsymbol{V} &= \partial_k \Big(\zeta \partial_t \boldsymbol{U} - \zeta \Delta \boldsymbol{U} - 2\nabla \zeta \cdot \nabla \boldsymbol{U} - (\Delta \zeta) \boldsymbol{U} \Big) \\
&\qquad - \partial_k (\partial_t \boldsymbol{z}_1 - \Delta \boldsymbol{z}_1 \Big) - \partial_k (\partial_t \boldsymbol{z}_2 - \Delta \boldsymbol{z}_2) \\
&= -\nabla \Pi + \partial_k (\, (P - P_{U^+_{2\rho}}) \nabla \zeta) - \partial_k \Big(2 \nabla \zeta \cdot \nabla \boldsymbol{U} + (\Delta \zeta) \boldsymbol{U} \Big) \\
&\qquad - \partial_k (\partial_t \boldsymbol{z}_1 - \Delta \boldsymbol{z}_1) - \partial_k (\partial_t \boldsymbol{z}_2 - \Delta \boldsymbol{z}_2) \\
&\qquad + \partial_k \Big(\zeta (\partial_n P) \nabla h - 2 \zeta \nabla h \cdot \nabla \partial_n \boldsymbol{U} + \zeta |\nabla h|^2 \partial_n \partial_n \boldsymbol{U} \\
&\qquad\qquad - \zeta (\Delta h) \partial_n \boldsymbol{U} + \zeta \boldsymbol{F} \Big).
\end{aligned}$$

Thus, $(\boldsymbol{V}, \Pi)$ solves the following Stokes system

$$\begin{aligned}
\operatorname{div} \boldsymbol{V} &= 0 && \text{in} \quad \mathbb{R}^n_+ \times (0,T), \\
\partial_t \boldsymbol{V} - \Delta \boldsymbol{V} &= -\nabla \Pi + \boldsymbol{G} && \text{in} \quad \mathbb{R}^n_+ \times (0,T), \\
\boldsymbol{V} &= 0 && \text{on} \quad \partial \mathbb{R}^n_+ \times (0,T),
\end{aligned}$$

where $\boldsymbol{G} = \boldsymbol{G}_1 + \ldots + \boldsymbol{G}_6$ with

$$\begin{aligned}
\boldsymbol{G}_1 &= \partial_{y_k} ((P - P_{U^+_{2\rho}}) \nabla \zeta), \\
\boldsymbol{G}_2 &= -\partial_k (2 \nabla \zeta \cdot \nabla \boldsymbol{U} + (\Delta \zeta) \boldsymbol{U}), \\
\boldsymbol{G}_3 &= -\partial_k (\partial_t \boldsymbol{z}_1 - \Delta \boldsymbol{z}_1 \Big), \\
\boldsymbol{G}_4 &= -\partial_k (\partial_t \boldsymbol{z}_2 - \Delta \boldsymbol{z}_2), \\
\boldsymbol{G}_5 &= \partial_k \Big(\zeta (\partial_n P) \nabla h - 2 \zeta \nabla h \cdot \nabla \partial_n \boldsymbol{U} + \zeta |\nabla h|^2 \partial_n \partial_n \boldsymbol{U} \Big), \\
\boldsymbol{G}_6 &= \partial_k (-\zeta (\Delta h) \partial_n \boldsymbol{U} + \zeta \boldsymbol{F}).
\end{aligned}$$

In what follows we shall establish some important estimates of $\boldsymbol{z}_1$ and $\boldsymbol{z}_2$, where we will make essential use of the properties of $\mathscr{B}$ (cf. Section 2). Starting with $\boldsymbol{z}_1$, we write $\boldsymbol{z}_1 = \boldsymbol{z}_{1,1} + \boldsymbol{z}_{1,2}$, where

$$\boldsymbol{z}_{1,1} = \mathscr{B}(\partial_n (\zeta \nabla h \cdot \boldsymbol{U})), \quad \boldsymbol{z}_{1,2} = -\mathscr{B}((\partial_n \zeta) \nabla h \cdot \boldsymbol{U})).$$

Let $t \in (0,T)$ be fixed. Using (2.5), (2.6) with $j = k, i = n$ and $f = \zeta \nabla h \cdot \boldsymbol{U}$, and observing $\partial_t \mathscr{B} = \mathscr{B} \partial_t$, we see that

$$\begin{aligned}
\|\partial_t \partial_k \boldsymbol{z}_1(t)\|_{\boldsymbol{L}^q(\mathbb{R}^n_+)} &\leq \|\partial_t \partial_k \boldsymbol{z}_{1,1}(t)\|_{\boldsymbol{L}^q(\mathbb{R}^n_+)} + \|\partial_t \partial_k \boldsymbol{z}_{1,2}(t)\|_{\boldsymbol{L}^q(\mathbb{R}^n_+)} \\
&\leq c \|\partial_t \partial_k (\zeta \nabla h \cdot \boldsymbol{U})(t)\|_{L^q(\mathbb{R}^n_+)} + c\rho^{-1} \|\partial_t \boldsymbol{U}(t)\|_{\boldsymbol{L}^q(\mathbb{R}^n_+)} \\
&\leq c\delta \|\partial_t \partial_k \widetilde{\boldsymbol{U}}(t)\|_{L^q(\mathbb{R}^n_+)} + c(\|h\|_{C^2} + \rho^{-1}) \|\partial_t \boldsymbol{U}(t)\|_{\boldsymbol{L}^q(U^+_{2\rho})}.
\end{aligned}$$

Taking the above inequality to the s-th power, and integrating the resulting equation in time over $(0,T)$, we get

$$\begin{aligned}
&\|\partial_t\partial_k \boldsymbol{z}_1\|_{L^s(0,T;\boldsymbol{L}^q(\mathbb{R}^n_+))} \\
&\quad\le c\delta\|\partial_t\partial_k\widetilde{\boldsymbol{U}}\|_{L^s(0,T;\boldsymbol{L}^q(\mathbb{R}^n_+))} + c(\|h\|_{C^2}+\rho^{-1})\|\partial_t \boldsymbol{U}\|_{L^s(0,T;\boldsymbol{L}^q(U_R^+))}.
\end{aligned} \tag{3.16}$$

On the other hand, using (2.5), (2.7) with $j=k, i=n$, and $f=\zeta\nabla h\cdot \boldsymbol{U}(t)$, we see that

$$\begin{aligned}
\|\nabla^2\partial_k \boldsymbol{z}_1(t)\|_{\boldsymbol{L}^q(\mathbb{R}^n_+)} \le\ & c\|\partial_n\nabla_*\nabla(\zeta\nabla h\cdot\boldsymbol{U})(t)\|_{L^q(\mathbb{R}^n_+)} + c\|\partial_n\partial_n\partial_k(\zeta\nabla h\cdot\boldsymbol{U})(t)\|_{L^q(\mathbb{R}^n_+)} \\
&+ c\rho^{-1}\|\nabla^2(\zeta\nabla h\cdot\boldsymbol{U})(t)\|_{L^q(\mathbb{R}^n_+)} + c\|\nabla^2((\partial_n\zeta)\nabla h\cdot\boldsymbol{U})(t)\|_{L^q(\mathbb{R}^n_+)}.
\end{aligned}$$

By means of product rule and Poincaré's inequality we find

$$\|\nabla^2\partial_k \boldsymbol{z}_1(t)\|_{\boldsymbol{L}^q(\mathbb{R}^n_+)} \le c\delta\|\nabla^2\nabla_*\widetilde{\boldsymbol{U}}(t)\|_{L^q(\mathbb{R}^n_+)} + c(\|h\|_{C^3}+\rho^{-1})\|\nabla^2\boldsymbol{U}(t)\|_{\boldsymbol{L}^q(U_R^+)}.$$

We now take the above inequality to the s-th power, integrating the result in time over $(0,T)$, we obtain

$$\begin{aligned}
\|\nabla^2\partial_k \boldsymbol{z}_1\|_{L^s(0,T;\boldsymbol{L}^q(\mathbb{R}^n_+))} \le\ & c\delta\|\nabla^2\nabla_*\widetilde{\boldsymbol{U}}\|_{L^s(0,T;L^q(\mathbb{R}^n_+))} \\
&+ c(\|h\|_{C^3}+\rho^{-1})\|\nabla^2\boldsymbol{U}\|_{L^s(0,T;\boldsymbol{L}^q(U_R^+))}.
\end{aligned} \tag{3.17}$$

By an analogous reasoning, making use of (2.5), and Poincare's inequality, we infer

$$\begin{aligned}
&\|\partial_t \boldsymbol{z}_2\|_{L^s(0,T;\boldsymbol{L}^q(\mathbb{R}^n_+))} + \|\nabla^2 \boldsymbol{z}_2\|_{L^s(0,T;\boldsymbol{L}^q(\mathbb{R}^n_+))} \\
&\qquad\le c\rho^{-1}\Big(\|\partial_t\boldsymbol{U}\|_{L^s(0,T;\boldsymbol{L}^q(U_R^+))} + \|\nabla^2\boldsymbol{U}\|_{L^s(0,T;\boldsymbol{L}^q(U_R^+))}\Big).
\end{aligned} \tag{3.18}$$

We are now in a position to estimate $\boldsymbol{G}_1,\dots,\boldsymbol{G}_6$. First by virtue of Poincaré's inequality we easily estimate

$$\|\boldsymbol{G}_1\|_{L^s(0,T;\boldsymbol{L}^q(\mathbb{R}^n_+))} \le c\rho^{-1}\|\nabla P\|_{L^s(0,T;\boldsymbol{L}^q(U_R^+))}.$$

Analogously,

$$\|\boldsymbol{G}_2\|_{L^s(0,T;\boldsymbol{L}^q(\mathbb{R}^n_+))} \le c\rho^{-1}\|\nabla^2\boldsymbol{U}\|_{L^s(0,T;\boldsymbol{L}^q(U_R^+))}.$$

Next, with the help of (3.16), (3.17), and (3.18) we see that

$$\begin{aligned}
&\|\boldsymbol{G}_3\|_{L^s(0,T;\boldsymbol{L}^q(\mathbb{R}^n_+))} + \|\boldsymbol{G}_4\|_{L^s(0,T;\boldsymbol{L}^q(\mathbb{R}^n_+))} \\
&\quad\le c\delta\Big(\|\partial_t\partial_k\widetilde{\boldsymbol{U}}\|_{L^s(0,T;\boldsymbol{L}^q(\mathbb{R}^n_+))} + \|\nabla^2\nabla_*\widetilde{\boldsymbol{U}}\|_{L^s(0,T;\boldsymbol{L}^q(\mathbb{R}^n_+))}\Big) \\
&\qquad + c(\|h\|_{C^3}+\rho^{-1})\Big(\|\partial_t\partial_k\widetilde{\boldsymbol{U}}\|_{L^s(0,T;\boldsymbol{L}^q(U_R^+))} + \|\nabla^2\nabla_*\widetilde{\boldsymbol{U}}\|_{L^s(0,T;\boldsymbol{L}^q(U_R^+))}\Big).
\end{aligned}$$

Then applying the product rule, and using Poincaré's inequality, we get

$$\begin{aligned}
\|\boldsymbol{G}_5\|_{L^s(0,T;\boldsymbol{L}^q(\mathbb{R}^n_+))} \le\ & c\delta\Big(\|\nabla\Pi\|_{L^s(0,T;\boldsymbol{L}^q(\mathbb{R}^n_+))} + \|\nabla_*\nabla^2\widetilde{\boldsymbol{U}}\|_{L^s(0,T;\boldsymbol{L}^q(\mathbb{R}^n_+))}\Big) \\
&+ c(\|h\|_{C^2}+\rho^{-1})\Big(\|\nabla P\|_{L^s(0,T;\boldsymbol{L}^q(U_R^+))} + \|\nabla^2\boldsymbol{U}\|_{L^s(0,T;\boldsymbol{L}^q(U_R^+))}\Big).
\end{aligned}$$

Finally, we estimate

$$\|\boldsymbol{G}_6\|_{L^s(0,T;\boldsymbol{L}^q(\mathbb{R}^n_+))} \le c(\|h\|_{C^3}+\rho^{-1})\Big(\|\nabla P\|_{L^s(0,T;\boldsymbol{L}^q(U_R^+))}+\|\nabla^2\boldsymbol{U}\|_{L^s(0,T;\boldsymbol{L}^q(U_R^+))}\Big). \\ + c\rho^{-1}\|\boldsymbol{F}\|_{L^s(0,T;\boldsymbol{L}^q(U_R^+))}+c\|\partial_k\boldsymbol{F}\|_{L^s(0,T;\boldsymbol{L}^q(U_R^+))}.$$

Appealing to Lemma 3.1 (cf. [7]) for the case $\Omega=\mathbb{R}^n_+$ using the above estimates for $\boldsymbol{G}_1,\dots,\boldsymbol{G}_6$, we obtain

$$\begin{aligned}
&\|\partial_t\boldsymbol{V}\|_{L^s(0,T;\boldsymbol{L}^q(\mathbb{R}^3_+))}+\|\nabla^2\boldsymbol{V}\|_{L^s(0,T;\boldsymbol{L}^q(\mathbb{R}^3_+))}+\|\nabla\Pi\|_{L^s(0,T;\boldsymbol{L}^q(\mathbb{R}^3_+))}\\
&\quad\le c\|\boldsymbol{G}_1+\ldots+\boldsymbol{G}_6\|_{L^s(0,T;\boldsymbol{L}^q(\mathbb{R}^n_+))}\\
&\quad\le c\delta\Big(\|\partial_t\nabla_*\widetilde{\boldsymbol{U}}\|_{L^s(0,T;\boldsymbol{L}^q(\mathbb{R}^3_+))}+\|\nabla^2\nabla_*\widetilde{\boldsymbol{U}}\|_{L^s(0,T;\boldsymbol{L}^q(\mathbb{R}^3_+))}+\|\nabla\Pi\|_{L^s(0,T;\boldsymbol{L}^q(\mathbb{R}^3_+))}\Big)\\
&\qquad+c(\|h\|_{C^3}+\rho^{-1})\Big(\|\partial_t\boldsymbol{U}\|_{L^s(0,T;\boldsymbol{L}^q(U_R^+))}+\|\nabla^2\boldsymbol{U}\|_{L^s(0,T;\boldsymbol{L}^q(U_R^+))}\\
&\qquad\qquad+\|\nabla P\|_{L^s(0,T;\boldsymbol{L}^q(U_R^+))}+\|\boldsymbol{F}\|_{L^s(0,T;\boldsymbol{W}^{1,q}(U_R^+))}\Big).
\end{aligned}$$

Recalling $\boldsymbol{V}=\partial_k(\widetilde{\boldsymbol{U}}-\boldsymbol{z}_1-\boldsymbol{z}_2)$, making use of (3.16), (3.17) and (3.18), from the last inequality we infer

$$\begin{aligned}
&\|\partial_t\nabla_*\widetilde{\boldsymbol{U}}\|_{L^s(0,T;\boldsymbol{L}^q(\mathbb{R}^3_+))}+\|\nabla^2\nabla_*\widetilde{\boldsymbol{U}}\|_{L^s(0,T;\boldsymbol{L}^q(\mathbb{R}^3_+))}+\|\nabla\Pi\|_{L^s(0,T;\boldsymbol{L}^q(\mathbb{R}^3_+))}\\
&\quad\le c_0\delta\Big(\|\partial_t\nabla_*\widetilde{\boldsymbol{U}}\|_{L^s(0,T;\boldsymbol{L}^q(\mathbb{R}^3_+))}+\|\nabla^2\nabla_*\widetilde{\boldsymbol{U}}\|_{L^s(0,T;\boldsymbol{L}^q(\mathbb{R}^3_+))}+\|\nabla\Pi\|_{L^s(0,T;\boldsymbol{L}^q(\mathbb{R}^3_+))}\Big)\\
&\qquad+c_1\Big(\|\partial_t\boldsymbol{U}\|_{L^s(0,T;\boldsymbol{L}^q(U_R^+))}+\|\nabla^2\boldsymbol{U}\|_{L^s(0,T;\boldsymbol{L}^q(U_R^+))}\\
&\qquad\qquad+\|\nabla P\|_{L^s(0,T;\boldsymbol{L}^q(U_R^+))}+\|\boldsymbol{F}\|_{L^s(0,T;\boldsymbol{W}^{1,q}(U_R^+))}\Big),
\end{aligned}\tag{3.19}$$

where $c_0=c_0(n,q,s)$ and $c_1=c_1(n,q,s,\|h\|_{C^3},\rho)$. On the other hand, recalling the definition of $\boldsymbol{U},P$, and $\boldsymbol{F}$, with the help of (3.10), (3.11), and (3.7) we find

$$\begin{aligned}
&\|\partial_t\boldsymbol{U}\|_{L^s(0,T;\boldsymbol{L}^q(U_R^+))}+\|\nabla^2\boldsymbol{U}\|_{L^s(0,T;\boldsymbol{L}^q(U_R^+))}\\
&\qquad+c\|\nabla P\|_{L^s(0,T;\boldsymbol{L}^q(U_R^+))}+\|\boldsymbol{F}\|_{L^s(0,T;\boldsymbol{W}^{1,q}(U_R^+))}\le c\|\boldsymbol{f}\|_{L^s(0,T;\boldsymbol{W}^{1,q}(\Omega))}
\end{aligned}\tag{3.20}$$

with a constant c depending on n,q,s and h. Now, in (3.19) we take $\delta=\frac{1}{2c_0}$ and estimate the right-hand side of (3.19) by the aid of (3.20). This leads to

$$\begin{aligned}
&\|\partial_t\nabla_*\widetilde{\boldsymbol{U}}\|_{L^s(0,T;\boldsymbol{L}^q(\mathbb{R}^3_+))}+\|\nabla^2\nabla_*\widetilde{\boldsymbol{U}}\|_{L^s(0,T;\boldsymbol{L}^q(\mathbb{R}^3_+))}+\|\nabla\Pi\|_{L^s(0,T;\boldsymbol{L}^q(\mathbb{R}^3_+))}\\
&\quad\le c_2\|\boldsymbol{f}\|_{L^s(0,T;\boldsymbol{W}^{1,q}(\Omega))},
\end{aligned}$$

where $c_2=c_2(n,q,s,\|h\|_{C^3},\rho)$.

By a standard iteration argument we obtain

$$\begin{aligned}
&\|\partial_t\nabla_*^k\boldsymbol{U}\|_{L^s(0,T;\boldsymbol{L}^q(U_\rho^+))}+\|\nabla^2\nabla_*^k\boldsymbol{U}\|_{L^s(0,T;\boldsymbol{L}^q(U_\rho^+))}+\|\nabla\nabla_*^kP\|_{L^s(0,T;\boldsymbol{L}^q(U_\rho^+))}\\
&\quad\le c\|\boldsymbol{f}\|_{L^s(0,T;\boldsymbol{W}^{1,q}(\Omega))},
\end{aligned}\tag{3.21}$$

where $c=$ const depending only on $n,q,s,k,\|h\|_{C^{k+2}}$ and ρ.

5° *Estimation of the full pressure gradient* Recalling that $\Delta_x p=0$, with the help of (3.10) we calculate

$$\begin{aligned}
0=\Delta_x p\circ\boldsymbol{\Phi}&=\Delta_y P-2\nabla h\cdot\nabla\partial_{y_n}P+|\nabla h|^2\partial_{y_n}\partial_{y_n}P-(\Delta h)\partial_{y_n}P\\
&=(1+|\nabla h|^2)\partial_n\partial_{y_n}P+\Delta_y'P-2\nabla h\cdot\nabla_*\partial_nP-(\Delta h)\partial_{y_n}P\ {}^{5)}
\end{aligned}$$

a. e. in U_R^+. Thus,

$$(1+|\nabla h|^2)\partial_{y_n}\partial_{y_n}P = -\Delta_y' P + 2\nabla h\cdot\nabla_*\partial_{y_n}P + (\Delta h)\partial_{y_n}P$$

a. e. in U_R^+. From this identity along with (3.21) with $k=1$ it follows that

$$\begin{aligned}\|\nabla_y^2 P\|_{L^s(0,T;\boldsymbol{L}^q(U_\rho^+))} &\le c\Big(\|\nabla_*\partial_{y_n}P\|_{L^s(0,T;\boldsymbol{L}^q(U_\rho^+))} + \|\nabla_y P\|_{L^s(0,T;\boldsymbol{L}^q(U_\rho^+))}\Big)\\ &\le c\|\boldsymbol{f}\|_{L^s(0,T;\boldsymbol{W}^{1,q}(\Omega))}.\end{aligned}$$

Choosing $\rho\in\left(0,\frac{R}{2}\right)$ sufficiently small, and applying the above argument k-times, we get

$$\|\nabla_y^{k+1}P\|_{L^s(0,T;\boldsymbol{L}^q(U_\rho^+))} \le c\|\boldsymbol{f}\|_{L^s(0,T;\boldsymbol{W}^{k,q}(\Omega))} \tag{3.22}$$

with a constant c depending on $n, q, s, k, \|h\|_{C^{k+2}}$, and ρ.

Finally a standard covering argument, together with (3.22), and (3.7) gives the estimate (1.5), which completes the proof of the Theorem 1. ■

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